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# Yang-Lee edge singularities in the large- $N$ limit

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We discuss the next-to-leading (but dominant for dimension less than six) corrections to the large- $N$  behavior of the magnetization at the Yang-Lee edge singularity. The  $N$  dependence of the corresponding amplitude is valid for all dimensions below six.

## I. INTRODUCTION

The role of zeros of the partition function was first pointed out by Yang and Lee.<sup>1</sup> The location of these zeros in the complex chemical-potential plane, for the case of a fluid, or in the complex magnetic-field plane, in the case of a spin system, plays a crucial role in the onset of phase transitions. For an Ising system, Yang and Lee showed that all of the zeros lie on the imaginary magnetic field  $\vec{H}$  axis<sup>2</sup> and as long as  $T > T_c$  it is generally believed that they do not come down to  $h=0$ . For a general  $O(N)$  spin system, one only knows that for  $T > T_c$  the partition function is free of zeros in a strip  $|\text{Im}H| < h_0$ .<sup>3</sup> We do, however, believe that the location of these zeros for the general case is, as in the Ising model, along the imaginary axis. The point  $H = ih_0(T)$  is a branch point of the partition function, referred to as the Yang-Lee edge singularity. For  $h = \text{Im}H > h_0$ , the density at the zeros is expected to vary as  $(\text{Im}H - h_0)^\sigma$  resulting in a magnetization<sup>4,5</sup>

$$\begin{aligned} M(ih) &= M_0(ih) + M_{\text{sing}}(ih), \\ M_{\text{sing}}(ih) &= A(h_0 - h)^\sigma, \end{aligned} \quad (1.1)$$

where  $M_0(ih)$  consists of a regular function of  $h$  and functions less singular than  $M_{\text{sing}}$ . This separation into regular and singular pieces is valid in the neighborhood of  $ih_0$ .

There is an accumulation of evidence<sup>4,5</sup> that the critical exponent  $\sigma$  is independent of temperature (as long as  $T > T_c$ ), the number of components of the  $O(N)$  spin system, and of course the details of the lattice. These results obtain in mean-field theory and large- $N$  calculations. It is the latter that lead to a seemingly paradoxical result. The large- $N$  calculation yields  $\sigma = \frac{1}{2}$  for all dimensions. However, for finite  $N$ , the  $\epsilon$  expansion for  $D = 6 - \epsilon$ ,<sup>5,6</sup> series expansions for  $D = 2$  and  $D = 3$ ,<sup>4</sup> and exact results for  $D = 0$  and  $D = 1$  yield an exponent  $\sigma$  decreasing from the value of  $\frac{1}{2}$  at  $D = 6$  down to  $\sigma = -\frac{1}{2}$  for  $D = 1$  and  $\sigma = -1$  at  $D = 0$ . (Details of the case  $D = 0$  will be

presented in Sec. IV.) The nonuniformity of the large- $N$  limit has been remarked on previously<sup>5</sup> and it is the purpose of this paper to clarify this point. We will show that below six dimensions, and for large, but finite  $N$ , the dominant singular part of the magnetization is expected to behave as

$$\frac{1}{N} M_{\text{sing}} = a(h_0 - h)^\sigma / N^c. \quad (1.2)$$

We obtain an expression for the exponent  $c$  valid for all dimensions less than six,

$$c = (2 - 4\sigma)/(6 - D). \quad (1.3)$$

This can be reconciled with the  $N = \infty$  result if one assumes for  $D < 6$ ,  $h_0 - h$ , and  $N^{-1}$  small, a scaling form

$$\frac{1}{N} M_{\text{sing}} = (h_0 - h)^{1/2} f((h_0 - h)N^\phi), \quad (1.4)$$

with  $f(x)$  behaving as a constant for  $x \rightarrow \infty$ , and as  $x^{\sigma-1/2}$  for  $x \rightarrow 0$ , with a crossover exponent

$$\phi = 4/(6 - D), \quad (1.5)$$

which is divergent when  $D$  reaches 6.

## II. LARGE- $N$ RESULTS

For large  $N$  the spin system will be studied by a saddle-point method. The partition function is

$$Z = \int \prod d\vec{s} \delta(\vec{s}^2 - N) \exp \left[ -\frac{\beta}{2} \int d^D x (\nabla \vec{s})^2 + \sqrt{N} \vec{H} \cdot \int d^D x \vec{s} \right], \quad (2.1)$$

where  $s$  and  $\vec{H}$  have been scaled as appropriate to this limit. We use continuum notation with a lattice spacing or momentum cutoff implied. The Dirac  $\delta$  functions are replaced by their integral representations

$$Z = \int \prod d\vec{s} \int_{-i\infty}^{i\infty} \prod dm^2 \exp \left[ -\int d^D x \left\{ \frac{\beta}{2} (\nabla \vec{s})^2 + \frac{m^2}{2} (\vec{s}^2 - N) - \sqrt{N} \vec{H} \cdot \vec{s} \right\} \right]. \quad (2.2)$$

The Gaussian integration over  $\vec{s}$  yields

$$Z = \int \prod dm^2 \exp \frac{N}{2} \left[ \int d^D x m^2 + \text{tr} \ln D(m) + H^2 \int d^D x d^D y D_{xy}(m) \right]. \quad (2.3)$$

The propagator  $D$  is given by

$$D_{xy}(m) = \langle x | (-\beta \nabla^2 + m^2)^{-1} | y \rangle. \quad (2.4)$$

The functional integration over  $m^2$  is performed by the saddle-point method, with a stationary point determined by

$$1 - D_{xx}(m^2) - H^2 \int d^D z d^D z' D_{zx}(m) D_{zx'}(m) = 0. \quad (2.5)$$

Assuming translational invariance, this equation reduces for a constant  $m^2$  to

$$1 - \int \frac{d^D k}{(2\pi)^D} \frac{1}{\beta k^2 + m^2} - \frac{H^2}{m^4} = 0. \quad (2.6)$$

(The zero magnetic-field critical temperature is given by  $\beta_c = \int d^D k / (2\pi)^D 1/k^2$ .) The behavior of the left-hand side of (2.6) as a function of  $m^2$  for various values of  $H = ih$  is plotted in Fig. 1. For  $T > T_c$  and  $h$  sufficiently small there are two solutions  $m^2 = m_1^2$  and  $m^2 = m_2^2$  with  $m_1^2 > m_2^2$ . The derivative of this same function indicates that  $m^2 = m_1^2$  is the correct saddle point. At the critical magnetic field  $h = h_0$ , the two solutions merge and for  $h > h_0$  they move into the complex  $m^2$  plane. As  $h$  approaches  $h_0$  from below, the sharpness of the saddle point decreases, and as the two points coalesce the derivative vanishes. For  $h \sim h_0$  and  $m^2$  close to the roots of Eq. (2.6), this function may be approximated by

$$a(m^2 - m_0^2)^2 + b(h^2 - h_0^2) = 0 \quad (2.7)$$

with  $a$  and  $b$  positive. The saddle-point contribution to the free energy per unit volume obtained from Eq. (2.3) is

$$F \sim N(h_0^2 - h^2)^{3/2} \quad (2.8)$$

yielding a singular contribution to the magnetization [cf. Eq. (1.1)]

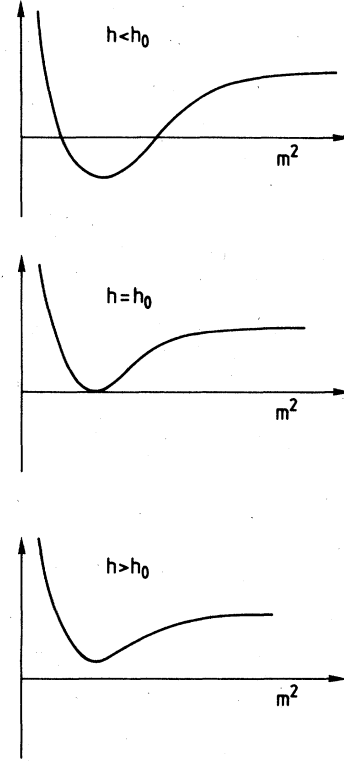


FIG. 1. Left-hand side of Eq. (2.3) as a function of  $m^2$  for three values of the magnetic field  $H = ih$ ,  $h_0$  is the critical field.

$$M_{\text{sing}}^{(0)} \sim N(h_0^2 - h^2)^{1/2}. \quad (2.9)$$

Thus, in the limit, first  $N \rightarrow \infty$ , then  $h \rightarrow h_0$ , the critical exponent  $\sigma$  is  $\frac{1}{2}$ . However, the above discussion makes the saddle-point method suspect just in the region  $h \rightarrow h_0$ . We should note in passing that as  $T$  approaches  $T_c$  both  $m_0$  and  $h_0$  approach zero.

The vanishing of the derivative reflects itself in the infrared behavior of the propagator of the fluctuations around the saddle point. With  $m^2(x) = m_1^2 - i\rho(x)$ , we find that up to terms cubic in  $\rho(x)$ ,

$$Z = Z(m_1^2) \int \prod d\rho \exp \left[ -\frac{N}{2} \int d^D x d^D y \rho(x) \Delta^{-1}(x, y) \rho(y) + \frac{i}{6} \int d^D x d^D y d^D z \rho(x) \rho(y) \rho(z) \Gamma(x, y, z) + \dots \right] \quad (2.10)$$

with

$$\frac{\ln Z(m_1^2)}{N\mathcal{V}} = \frac{m_1^2}{2} - \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \ln(\beta k^2 + m_1^2) - \frac{h^2}{2m_1^2}, \quad (2.11a)$$

$$\Delta^{-1}(x, y) = [D_{xy}(m_1)]^2 - \frac{2h^2}{m_1^4} D_{xy}(m_1), \quad (2.11b)$$

$$\Gamma(x, y, z) = 2D_{xy}(m_1)D_{yz}(m_1)D_{zx}(m_1) - \frac{2h^2}{m_1^4} \{ [D_{xz}(m_1)D_{zy}(m_1)] + \text{c.p.} \}, \quad (2.11c)$$

where again c.p. stands for terms obtained under cyclic permutations of  $x, y, z$ . The Fourier transform of the inverse propagator is

$$\Delta^{-1}(x, y) = \int \frac{d^D l}{(2\pi)^D} \tilde{\Delta}^{-1}(l) e^{il \cdot (x - y)}, \quad (2.12)$$

$$\tilde{\Delta}^{-1}(l) = \int \frac{d^D k}{(2\pi)^D} \left[ \frac{1}{(\beta k^2 + m_1^2)[\beta(k-l)^2 + m_1^2]} \right] - \frac{2h^2}{m_1^4} \frac{1}{\beta l^2 + m_1^2}.$$

We note that  $\tilde{\Delta}^{-1}(0)$  is just the derivative with respect to  $m^2$  of the left-hand side of Eq. (2.6) and vanishes at  $h=h_0$ .

Thus, at the critical magnetic field we are faced with a massless  $i\varphi^3$  field theory, where the cubic couplings are nonlocal. The range of nonlocality is  $1/m_1$ . As long as  $T > T_c$ ,  $m_1 \neq 0$  and we may view  $1/m_1$  as a new lattice constant. In the critical region we are interested in large distances, larger than  $1/m_1$ . In this regime the field theory can be considered as local. Above six dimensions the masslessness of the theory is innocuous<sup>7</sup> and we recover the value  $\sigma = \frac{1}{2}$ ; below six dimensions, the infrared behavior is crucial and  $\sigma$  will deviate from this value. The critical Hamiltonian will be independent of  $T$  and  $N$ , reflecting itself in the fact that  $\sigma$  is also independent of these parameters. Details will be presented in the next section.

### III. CONTRIBUTIONS OF FLUCTUATIONS TO THE MAGNETIZATION

The effective-field theory for the long-wavelength fluctuations around the saddle point is obtained from Eqs. (2.10)–(2.12). The action is

$$S = N \int d^D x \left[ \frac{1}{2} (\nabla \rho)^2 + \frac{1}{2} \mu^2 \rho^2 + \frac{i}{6} \lambda \rho^3 + \dots \right] \quad (3.1)$$

with

$$\mu^2 \sim (h_0^2 - h^2)^{1/2} \quad (3.2)$$

and the constant  $\lambda$  has dimension  $(6-D)/2$ . The factor  $N$  in Eq. (3.1) may be removed by rescaling the field  $\rho = \varphi \sqrt{N}$ . In terms of  $\varphi$  the action is

$$S = \int d^D x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \mu^2 \varphi^2 + \frac{i\lambda}{6\sqrt{N}} \varphi^3 + \dots \right]. \quad (3.3)$$

The dependence of this action on the magnetic field is contained in the effective mass  $\mu$ . Thus, the contribution of the fluctuation to the magnetization is

$$M_{\text{sing}} \sim \frac{1}{\mu^2} \langle \varphi^2 \rangle. \quad (3.4)$$

Using the equations of motion we find that

$$M_{\text{sing}} \sim \frac{\sqrt{N}}{\lambda} \langle \varphi \rangle. \quad (3.5)$$

For  $D < 6$ , the renormalization-group equations, with the  $\beta$  function and anomalous dimensions calculated by the  $\epsilon$  expansion ( $\epsilon = 6 - D$ ) may be applied to either Eqs. (3.4) or (3.5) (Refs. 5–7) yielding

$$M_{\text{sing}} = A(N) (\mu^2)^{2\sigma}, \quad (3.6)$$

$$\sigma = \frac{1}{2} - \frac{1}{12} \epsilon - \frac{79}{3888} \epsilon^2 + \left[ \frac{\xi(3)}{81} - \frac{10445}{1259712} \right] \epsilon^3 + \dots$$

We want to obtain the  $N$  dependence, for large  $N$ , of the coefficient  $A$  in Eq. (3.6).

For  $h \sim h_0$  the only dimensional parameters entering the theory are  $\mu^2$  and  $\lambda/\sqrt{N}$ . From Eqs. (3.5) and (3.6),

and remembering that the dimension of  $\varphi$  is  $(D-2)/2$ , we obtain

$$M_{\text{sing}} \sim \frac{\sqrt{N}}{\lambda} \left[ \frac{\lambda}{\sqrt{N}} \right]^{(D-2)/(6-D)} \left[ \mu^2 \left[ \frac{\lambda}{\sqrt{N}} \right]^{-4/(6-D)} \right]^{2\sigma} \quad (3.7)$$

or

$$\frac{1}{N} M_{\text{sing}} = a (h_0^2 - h^2)^\sigma / (N^{(2-4\sigma)/(6-D)}) \quad (3.8)$$

with  $a$  being  $N$  independent. As discussed in Sec. I for  $D < 6$ , the more singular behavior in  $h$  is subdominant in  $N$ . An identical result may be obtained using Eq. (3.4) rather than Eq. (3.5).

The arguments leading to Eq. (3.7) form a special case of the general ones developed by Parisi<sup>8</sup> to obtain the bare coupling-constant dependence. At first glance, we expect the above results to be valid for  $4 < D < 6$ . Below four dimensions, other operators may become relevant and spoil the simple scaling ideas used to obtain Eq. (3.7). We shall, however, in Sec. IV confirm (3.8) in low dimension, so that we expect it to be valid for all dimensions less than six.

### IV. ZERO- AND ONE-DIMENSIONAL CASES

In zero and in one dimensions the critical exponent  $\sigma$  is known exactly. For  $D=1$ , the Ising model in an external magnetic field can be solved yielding  $\sigma = -\frac{1}{2}$ ; for  $D=0$ , as we shall show below,  $\sigma = -1$ . It is instructive, however, to obtain these results within the large- $N$  framework.

#### A. Zero dimensions

Zero dimensions means that we are dealing with a single site. The partition function for a spin in an imaginary magnetic field is

$$Z = \frac{1}{S_N} \int d\hat{s} e^{iNh \cdot \hat{s}/2} \sim h^{-N/2} J_{N/2} \left[ \frac{Nh}{2} \right], \quad (4.1)$$

where  $\hat{s}$  is a unit  $(N+2)$ -dimensional vector and  $S_N$  is the area of the unit  $N$ -dimensional sphere. The magnetic field is for subsequent convenience chosen to be  $Nh/2$ . The Bessel function has discrete real zeros. (This property of Bessel functions is the analog for  $D=0$  of the Lee-Yang<sup>2</sup> theorem on the zeros of the partition function of an Ising model.) The magnetization obtained from Eq. (4.1) has discrete poles. Thus in  $D=0$ ,  $\sigma = -1$ .

For large  $N$  the integral

$$Z = \int_{-1}^1 dx \exp \frac{N}{2} [ihx + \ln(1-x^2)] \quad (4.2)$$

can be evaluated by the saddle-point method. The position of the saddle point is given by

$$x_0 = \frac{i}{h} [1 - (1-h^2)^{1/2}] \quad (4.3)$$

and its contribution yields a magnetization with a critical field  $h_0=1$  and  $\sigma = \frac{1}{2}$ . The  $N$  dependence of the relative

coefficients of the above contribution to the magnetization and that due to the simple zeros of the Bessel function is in accord with our previous discussion.

The saddle point gives the correct asymptotic behavior in  $N$  of the Bessel function as long as  $h \neq 1$ . For  $h \sim 1$ , we again find that this saddle point disappears and higher terms in the expansion must be retained. Keeping up to terms cubic in  $x - x_0$  we find<sup>9</sup> (for  $h \sim 1$ )

$$Z \sim \int_{-\infty}^{+\infty} dx \exp \frac{-iN}{12} [x^3 - 3i(1-h^2)x] \\ = \frac{2\pi}{(N/4)^{1/3}} \text{Ai} \left[ \left[ \frac{N}{4} \right]^{2/3} (1-h^2) \right]. \quad (4.4)$$

This function has zeros at  $h_i^2 = 1 + x_i(4/N)^{2/3}$ , with the differences of the  $x_i$ 's being of order 1. The density of zeros varies as  $(h-1)^{3/2}$ , which agrees with the large- $N$  behavior of the free energy.

Correspondingly, for  $N$  large and  $h \rightarrow h_0 = 1 + \frac{1}{2}x_1(4/N)^{2/3}$ , we can write a scaling form of the singular part of the magnetization

$$\frac{1}{N} M_{\text{sing}} = \left[ \frac{h_0 - h}{2} \right]^{1/2} \left[ -\frac{1}{\sqrt{x}} \frac{\text{Ai}'(x - x_1)}{\text{Ai}(x - x_1)} \right], \\ x = 2 \left[ \frac{N}{4} \right]^{2/3} (h_0 - h) \quad (4.5)$$

where  $-x_1$  is the first negative zero of the Airy function. The quantity in the second set of parentheses has the expected behavior, going to  $-x^{-3/2}$  for small  $x$ , and to 1 for  $x \rightarrow \infty$ . The exponent  $\phi = \frac{2}{3}$  agrees with (1.5).

### B. One dimension

A one-dimensional field theory corresponding to Eq. (3.3) may be considered as a problem in quantum mechanics with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + \frac{1}{2}\mu^2 q^2 + \frac{i\lambda}{6\sqrt{N}} q^3 + \dots \quad (4.6)$$

The free energy is the "ground-state energy" of this system. In order to evaluate the relevant part of the magnetization we split  $\mathcal{H}$  into two parts:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

$$\mathcal{H}_0 = \frac{p^2}{2} + \frac{i\lambda}{6\sqrt{N}} q^3 + \dots, \quad (4.7)$$

$$\mathcal{H}_1 = \frac{1}{2}\mu^2 q^2,$$

and treat  $\mathcal{H}_1$  perturbatively. After rescaling to a variable  $x = (\lambda\sqrt{N})^{1/3} q$  we note that

$$\langle \mathcal{H}_1 \rangle \sim \left[ \frac{\lambda}{\sqrt{N}} \right]^{-2/5} \mu^2, \quad \frac{1}{N} M_{\text{sing}} \sim \frac{1}{N^{4/5}} \frac{1}{(h_0 - h)^{1/2}}. \quad (4.8)$$

We find  $\sigma = -\frac{1}{2}$  and  $c = \phi = 0.8$ . This should come as no surprise as we have used the same scaling arguments that lead to Eq. (3.7). The value for the crossover exponent  $\phi$  can perhaps be identified with the scaling exponent  $\xi = 0.73 \pm 0.03$  obtained<sup>10</sup> using transfer matrix methods.

This example also shows us, that even though operators  $\varphi^n$  with  $n > 3$  are relevant in lower dimensions it is still the  $\varphi^3$  operator that governs the behavior of the singular part of the magnetization. Had we been interested in expectation values of higher powers of  $\mathcal{H}_1$ , which would appear in higher orders of perturbation, then other terms in the potential of Eq. (4.8) would come into play. These would, however, contribute to less singular behavior of the magnetization.

## V. CONCLUSIONS

For  $D < 6$ , the most singular behavior of the magnetization at the Yang-Lee edge singularity is subdominant in  $N$  to the large- $N$  results. The behavior in  $N$  of the relative coefficient of these two terms has been evaluated for all dimensions. We argued that the  $\varphi^3$  term in the action of the fluctuations around the large- $N$  limit determines the behavior of the most singular term, not only for  $D \sim 6$ , but for all  $D$  down to  $D = 0$ .

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